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Congruence Properties of Apéry Numbers

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Apéry introduced a recurrence relation for a proof of the irrationality of $\zeta(3)$. Let a_n $(n \geqslant 0)$ satisfy the relation $n^3 a_n - (34n^3 - 51n^2 + 27n - 5)a_{n-1} + (n-1)^3 a_{n-2} = 0$. Which values of a_0 and a_1 cause each a_n to be an integer? This question is answered and some congruence properties of the a_n are given.

For his proof of the irrationality of $\zeta(3) = \sum n^{-3}$ (see [1]), Apéry studied the recurrence relation

$$n^3 a_n - (34n^3 - 51n^2 + 27n - 5)a_{n-1} + (n-1)^3 a_{n-2} = 0 \qquad (n \geqslant 2). \tag{1}$$

If A_n $(n \geqslant 0)$ denotes the solution of (1) defined by the initial conditions $A_0 = 1$ and $A_1 = 5$, then Apéry showed that

$$A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2. \tag{2}$$

Chowla *et al.* [2] proved that

$$\begin{aligned} A_n &\equiv 1 && \bmod 2 && \text{for all } n \geqslant 0, \\ A_p &\equiv 5 && \bmod p^2 && \text{for all primes } p, \end{aligned}$$

and conjectured that

$$\begin{aligned} A_n &\equiv 1 + 4n && \bmod 8 && \text{for all } n \geqslant 0, \\ A_n &\equiv (-1)^n && \bmod 3 && \text{for all } n \geqslant 0, \\ A_p &\equiv 5 && \bmod p^3 && \text{for all primes } p > 3, \\ A_p &\equiv 0 && \bmod 5 && \text{for odd primes } p. \end{aligned}$$

They also posed a question: which values of a_0 and a_1 in (1) cause each a_n to be an integer?

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In this note we shall answer this question and give some congruences which include the above conjectures.

THEOREM 1. *Let a_n be numbers satisfying* (1) *with integers a_0 and a_1. Then each a_n is an integer if and only if $a_1 = 5a_0$.*

Proof. Let $f(x) = 5 - 27x + 51x^2 - 34x^3$. For $1 \leqslant i \leqslant j$, we define a matrix $\mathscr{D}(i, j)$ by

$$\begin{pmatrix} f(i) & i^3 & & & & \\ i^3 & f(i+1) & (i+1)^3 & & & \\ & (i+1)^3 & f(i+2) & & 0 & \\ & \vdots & & \ddots & \vdots & \\ & 0 & & & f(j-1) & (j-1)^3 \\ & & & & (j-1)^3 & f(j) \end{pmatrix}$$

Let p be an odd prime and set $p^* = p$ if $p > 3$, $p^* = 9$ if $p = 3$. Since $f(x) + f(1-x) = 0$ and $f(p+k) \equiv f(k) \bmod p^*$, we have

$$f(i) + f(j) \equiv 0 \qquad \bmod p^* \qquad \text{if} \quad i + j = p + 1. \tag{3}$$

Letting

$$\mathscr{W} = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix},$$

from (3), we have

$$\mathscr{W}\mathscr{D}(i, j)\mathscr{W} \equiv -\mathscr{D}(i, j) \qquad \bmod p^* \qquad \text{if} \quad i + j = p + 1.$$

Putting $D(i, j) = \det \mathscr{D}(i, j)$, we have

$$D(i, j) \equiv 0 \qquad \bmod p^* \qquad \text{if} \quad i + j = p + 1. \tag{4}$$

By the definition of $\mathscr{D}(i, j)$ and (4),

$$\begin{aligned} D(2, p) &= f(p)\, D(2, p-1) - (p-1)^6\, D(2, p-2) \\ &\equiv 1^3(p-1)^3\, D(2, p-2) && \bmod p^* \\ &= 1^3(p-1)^3\{f(2)\, D(3, p-2) - 2^6\, D(4, p-2)\} \\ &\equiv 1^3(p-1)^3 2^3 (p-2)^3\, D(4, p-2) && \bmod p^* \\ &\equiv \cdots \equiv \{(p-1)!\}^3 && \bmod p^*, \\ D(2, p) &\equiv -1 && \bmod p^*. \end{aligned} \tag{5}$$

Consider numbers a_n satisfying (1). Let $t \geqslant 0$. Putting

$$\mathscr{A}_t = (a_{pt+1}, a_{pt+2}, \ldots, a_{pt+p-1}),$$
$$\mathscr{B}_t = (-(pt+1)^3 a_{pt}, 0, \ldots, 0, -(pt+p)^3 a_{pt+p}),$$

from (1) we have

$$\mathscr{A}_t \mathscr{D}(pt+2, pt+p) = \mathscr{B}_t. \tag{6}$$

Suppose that each a_n is an integer. Since $\mathscr{D}(pt+2, pt+p) \equiv \mathscr{D}(2, p) \bmod p^*$,

$$\mathscr{A}_t \mathscr{D}(2, p) \equiv \mathscr{B}_t \qquad \bmod p^*.$$

From (5), there is the inverse $\mathscr{D}_p^*$ of $\mathscr{D}(2, p) \bmod p^*$. Hence

$$\mathscr{A}_t \equiv \mathscr{B}_t \mathscr{D}_p^* \qquad \bmod p^*.$$

Let $(-\lambda_p(1), \ldots, -\lambda_p(p-1))$ be the first row of $\mathscr{D}_p^*$. Then

$$a_{pt+j} \equiv \lambda_p(j)\, a_{pt} \qquad \bmod p^* \tag{7}$$

for $j = 0, 1, \ldots, p-1$, putting $\lambda_p(0) = 1$. From (4),

$$0 \equiv D(1, p) = f(1)\, D(2, p) - 1^6 \cdot D(3, p) \qquad \bmod p^*,$$
$$D(3, p) \equiv 5 \qquad \bmod p^*,$$
$$-\lambda_p(1) \equiv D(3, p)\, D(2, p)^{-1} \equiv -5 \qquad \bmod p^*.$$

Hence $a_1 \equiv \lambda_p(1) a_0 \equiv 5a_0 \bmod p^*$. Taking a prime p such that $p > |a_1 - 5a_0|$, this shows that $a_1 = 5a_0$. Conversely, suppose that $a_1 = 5a_0$. Then each a_n is an integer from Apéry's result.

Remark (due to Zagier). Let B_n be the solution of (1) with $B_0 = 0$, $B_1 = 1$. The main ingredients in Apéry's proof of the irrationality of $\zeta(3)$ were the assertions

$$12 N_n^3 B_n \in \mathbb{Z}, \qquad (N_n = \text{l.c.m.}\,\{1, 2, \ldots, n\}), \tag{8}$$
$$B_n / A_n = \zeta(3)/6 + O((1+\sqrt{2})^{-8n}). \tag{9}$$

In fact the 12 can be omitted in (8) (though this is not quite trivial), but apart from that, our argument shows that (8) is essentially best possible. Indeed, taking $a_n = B_n$ and $t = 0$ in (6) and observing that $B_1, \ldots, B_{p-1}$ are p-integral and $B_0 = 0$, we deduce from (6) and (5) the statement

$$p^3 B_p \in \mathbb{Z}_p, \qquad p^2 B_p \notin \mathbb{Z}_p.$$

Here $\mathbb{Z}_p$ denotes the ring of p-integers. Note that this statement actually implies Theorem 1, since any solution of (1) is a linear combination of $\{A_n\}$ and $\{B_n\}$.

From now on we consider the numbers a_n satisfying (1) with $a_0 = 1$ and $a_1 = 5$, i.e., the numbers A_n defined by (2).

PROPOSITION 1. *We have*

$$A_n \equiv 1 + 4n \mod 8 \qquad \textit{for all} \quad n \geqslant 0.$$

Proof. By $[\ ]$ we denote the greatest integer function. Let $k > 0$ and e be the integer such that $2^e \mid \binom{2k}{k}$ and $2^{e+1} \nmid \binom{2k}{k}$. Then

$$e = \sum_{r=1}^{\infty} \left[\frac{2k}{2^r}\right] - 2\sum_{r=1}^{\infty} \left[\frac{k}{2^r}\right] = k - \sum_{r=1}^{\infty} \left[\frac{k}{2^r}\right] > k - \sum_{r=1}^{\infty} \frac{k}{2^r} = 0.$$

Hence $\binom{2k}{k}$ is even if $k > 0$. Thus $\binom{n+k}{k}\binom{n}{k} = \binom{n+k}{2k}\binom{2k}{k}$ is even. Hence $\binom{n+k}{k}\binom{n}{k} \equiv (-1)^k \binom{n+k}{k}\binom{n}{k} \bmod 4$ and

$$A_n \equiv 1 + \sum_{k=1}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2 \equiv 1 + \left\{\sum_{k=1}^{n} \binom{n+k}{k}\binom{n}{k}\right\}^2 \qquad \bmod 8$$

$$\equiv 1 + \left\{\sum_{k=1}^{n} (-1)^k \binom{n+k}{k}\binom{n}{k}\right\}^2 \qquad \bmod 8.$$

By the Lemma in [2], $T(n) = \sum_{k=0}^{n} (-1)^k \binom{n+k}{k}\binom{n}{k} = (-1)^n$. Hence $A_n \equiv 1 + ((-1)^n - 1)^2 \equiv 1 + 4n \bmod 8$.

LEMMA. *We have*

$$\binom{pm}{pk} \equiv \binom{m}{k} \mod p^3 \qquad \textit{for all primes } p \geqslant 5,$$

$$\binom{pm}{pk} \equiv \binom{m}{k} \mod p^2 \qquad \textit{for all primes } p.$$

Proof. Let p be a prime greater than 3.

$$\binom{pm}{pk} = \prod_{i=1}^{pk} \frac{p(m-k)+i}{i} = \binom{m}{k} \prod_{h=0}^{k-1} u_h, \qquad u_h = \prod_{j=1}^{p-1} \frac{p(m-k+h)+j}{ph+j}.$$

$$\prod_{j=1}^{p-1} (pt+j) = \left(\prod_{j=1}^{p-1} j\right)\left(\prod_{j=1}^{p-1} \left(1 + \frac{pt}{j}\right)\right)$$

$$\equiv (p-1)! \left(1 + pt \sum_{j=1}^{p-1} \frac{1}{j} + p^2t^2 \sum_{0<i<j<p} \frac{1}{ij}\right) \qquad \bmod p^3.$$

Since $\sum_{j=1}^{p-1} 1/j \equiv 0 \bmod p^2$ and $\sum_{0<i<j<p} (1/ij) \equiv 0 \bmod p$, $\prod_{j=1}^{p-1}(pt+j) \equiv (p-1)! \bmod p^3$. Hence $u_h \equiv 1 \bmod p^3$, i.e., $\binom{pm}{pk} \equiv \binom{m}{k} \bmod p^3$. If $p=3$, then $u_h \equiv 1 \bmod 9$. Thus $\binom{3m}{3k} \equiv \binom{m}{k} \bmod 3^2$. If $p=2$, then $u_h \equiv 1 + 2(m-k) \bmod 4$. Hence $\binom{m}{k} u_0 u_1 \cdots u_{k-1} \equiv \binom{m}{k}(1+2(m-k))^k \equiv \binom{m}{k}(1+2k(m-k)) \equiv \binom{m}{k} + 2k(m-k)\binom{m}{k} \bmod 4$. If $k=0$ or m, then $2k(m-k)\binom{m}{k} \equiv 0 \bmod 4$. If $0<k<m$, then $2k(m-k)\binom{m}{k} = 2m(m-1)\binom{m-2}{k-1} \equiv 0 \bmod 4$. Thus $\binom{2m}{2k} \equiv \binom{m}{k} \bmod 2^2$.

PROPOSITION 2. *We have*

$$A_{pm} \equiv A_m \mod p^3 \qquad \textit{for all primes } p \geqslant 5,$$

$$A_{pm} \equiv A_m \mod p^2 \qquad \textit{for all primes } p.$$

Proof. This is trivial when $m=0$. Let $m>0$. Put $e=3$ or 2 according as $p>3$ or not. By the Lemma,

$$\begin{aligned} A_{pm} &= \sum_{i=0}^{m} \binom{pm}{pi}^2 \binom{p(m+i)}{pi}^2 \\ &\quad + \sum_{t=0}^{m-1} \sum_{i=1}^{p-1} \binom{pm}{pt+i}^2 \binom{pm+pt+i}{pt+i}^2 \\ &\equiv A_m + \sum_{t=0}^{m-1} \sum_{i=1}^{p-1} \binom{pm}{pt+i}^2 \binom{pm+pt+i}{pt+i}^2 \\ &\qquad \bmod p^e; \end{aligned}$$

$$\binom{pm+pt+i}{pt+i} = \prod_{j=1}^{pt+i} \frac{pm+j}{j} \equiv \prod_{i=1}^{t} \frac{m+i}{i} = \binom{m+t}{t} \qquad \bmod p;$$

$$\begin{aligned} (p(m-t)-i)\binom{pm}{pt+i} &= pm \prod_{s=1}^{pt+i} \frac{pm-s}{s} \\ &= pm \prod_{h=0}^{t-1} \left(\prod_{j=1}^{p-1} \frac{p(m-h)-j}{ph+j} \right) \\ &\quad \times \prod_{h=1}^{t} \frac{p(m-h)}{ph} \cdot \prod_{j=1}^{i} \frac{p(m-t)-j}{pt+j} \\ &\equiv pm(-1)^{(p-1)t+i} \binom{m-1}{t} \qquad \bmod p^2; \end{aligned}$$

$$\binom{pm}{pt+i}^2 \equiv p^2 m^2 \binom{m-1}{t}^2 \frac{1}{i^2} \qquad \bmod p^3;$$

$$\binom{pm}{pt+i}^2 \equiv 0 \qquad \text{mod } p^2;$$

$$\sum_{i=1}^{p-1} \binom{pm}{pt+i}^2 \binom{pm+pt+i}{pt+i}^2 \equiv \sum_{i=1}^{p-1} \binom{pm}{pt+i}^2 \binom{m+t}{t}^2 \qquad \text{mod } p^3$$

$$\equiv p^2 m^2 \binom{m-1}{t}^2 \binom{m+t}{t}^2 \sum_{i=1}^{p-1} \frac{1}{i^2} \qquad \text{mod } p^3$$

$$\equiv 0 \qquad \text{mod } p^e.$$

COROLLARY. *We have*

$$A_p \equiv 5 \quad \text{mod } p^3 \qquad \textit{for all primes } p \geqslant 5,$$
$$A_p \equiv 5 \quad \text{mod } p^2 \qquad \textit{for all primes } p.$$

THEOREM 2. *Let p be an odd prime and $n = \sum_{j=0}^{\infty} e_j p^j$ with $e_j \in \{0, 1, ..., p-1\}$. Then*

$$A_n \equiv \prod_{j=0}^{\infty} \lambda_p(e_j) \qquad \text{mod } p^*.$$

Proof. This follows immediately from (7) and Proposition 2.

COROLLARY 1. *We have*

$$A_n \equiv (-1)^n \quad \text{mod } 3 \qquad \text{for all } n \geqslant 0.$$

Proof. Let $n = \sum_{j=0}^{\infty} e_j 3^j$ with $e_j \in \{0, 1, 2\}$. Then $n \equiv \sum_{j=0}^{\infty} e_j \bmod 2$. We have $\lambda_3(0) = \lambda_3(2) = 1$ and $\lambda_3(1) = 5$. By Theorem 2,

$$A_n \equiv \prod_{j=0}^{\infty} \lambda_3(e_j) \equiv \prod_{j=0}^{\infty} (-1)^{e_j} = (-1)^{\Sigma e_j} \equiv (-1)^n \text{ mod } 3.$$

COROLLARY 2. *We have*

$$A_n \equiv \text{mod } 5 \qquad \textit{for all odd } n.$$

Proof. We have $\lambda_5(0) = \lambda_5(4) = 1$, $\lambda_5(1) = \lambda_5(3) = 0$ and $\lambda_5(2) = 3$. Let $n = \sum_{j=0}^{\infty} e_j 5^j$ with $e_j \in \{0, 1, 2, 3, 4\}$. Since n is odd, there is at least one j such that $e_j \equiv 1 \bmod 2$, i.e., such that $\lambda_5(e_j) = 0$. Hence by Theorem 2 we have $A_n \equiv \prod_{j=0}^{\infty} \lambda_5(e_j) \equiv 0 \bmod 5$.

Remark. By Theorem 2 we can find the residue of A_n mod p for any n. For example, $A_{10000} \equiv 3 \cdot 1 \cdot 5 \cdot 5 \cdot 3 \equiv 1 \mod 7$ since $10000 = 4 + 0 \cdot 7 + 1 \cdot 7^2 + 1 \cdot 7^3 + 4 \cdot 7^4$ and $\lambda_7(0) = \lambda_7(6) = 1$, $\lambda_7(1) = \lambda_7(5) = 5$ and $\lambda_7(2) = \lambda_7(3) = \lambda_7(4) = 3$. Similarly, $A_{5t+1} \equiv A_{5t+3} \equiv 0 \mod 5$, $A_{11t+5} \equiv 0 \mod 11$, $A_{17t+13} \equiv A_{17t+13} \equiv 0 \mod 17, \ldots$ since $\lambda_5(1) = \lambda_5(3) = 0$, $\lambda_{11}(5) = 0$, $\lambda_{17}(3) = \lambda_{17}(13) = 0, \ldots$.

PROPOSITION 3. *We have*

$$A_{p-1} \equiv 1 \mod p^3 \qquad \textit{for all primes } p \geqslant 5,$$
$$A_{p-1} \equiv 1 \mod p^2 \qquad \textit{for all primes } p.$$

Proof. This is easy when $p = 2$ or $p = 3$. Let $p > 3$. For $0 < k < p$,

$$\binom{p-1+k}{k} = \frac{p}{k} \prod_{i=1}^{k-1} \frac{p+i}{i} \equiv \frac{p}{k} \mod p^2,$$
$$\binom{p-1+k}{k}^2 \equiv \left(\frac{p}{k}\right)^2 \mod p^3 \qquad \text{and} \qquad \binom{p-1+k}{k}^2 \equiv 0 \mod p^2;$$
$$\binom{p-1}{k} = \prod_{i=1}^{k} \frac{p-i}{i} \equiv (-1)^k \mod p, \qquad \binom{p-1}{k}^2 \equiv 1 \mod p.$$

Hence

$$A_{p-1} = 1 + \sum_{k=1}^{p-1} \binom{p-1}{k}^2 \binom{p-1+k}{k}^2$$
$$\equiv 1 + \sum_{k=1}^{p-1} \frac{p^2}{k^2} \equiv 1 + p^2 \sum_{k=1}^{p-1} \frac{1}{k^2} \equiv 1 \mod p^3.$$

By (2), Proposition 2, and Proposition 3, we have the congruences

$$A_{p+1} \equiv 25 + 60p \mod p^3 \qquad \text{for all primes } p \geqslant 5,$$
$$A_{p+2} \equiv 365 + 1050p + 360p^2 \mod p^3 \qquad \text{for all primes } p \geqslant 5,$$
$$A_{p-2} \equiv 5 - 12p \mod p^3 \qquad \text{for all primes } p \geqslant 5,$$

and so forth.

PROPOSITION 4. *Let* $i \geqslant 0, j \geqslant 0, t \geqslant 0$ *and* $i + j = p - 1$. *Then*

$$\lambda_p(i) \equiv \lambda_p(j) \qquad \mod p^*,$$
$$A_{pt+i} \equiv A_{pt+j} \qquad \mod p^*.$$

Proof. We proceed by induction on i to show that $A_i \equiv A_{p-i-1} \bmod p^*$. It has already been shown that $A_0 \equiv A_{p-1} \equiv 1 \bmod p^*$ and $A_1 \equiv A_{p-2} \equiv 5 \bmod p^*$. Let $p-1 > i > 0$. Then by (1), (3), and the induction hypothesis,

$$\begin{aligned}(i+1)^3 A_{i+1} &\equiv -f(i+1)A_i - i^3 A_{i-1} \equiv f(p-i)A_i + (p-i)^3 A_{i-1}\\ &\equiv f(p-i)A_{p-i-1} + (p-i)^3 A_{p-i} \equiv -(p-i-1)^3 A_{p-i-2}\\ &\equiv (i+1)^3 A_{p-i-2} \qquad \bmod p^*.\end{aligned}$$

TABLE I

Values of $\lambda_p(j) \bmod p^*$

j \ p	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	5	0	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
2		3	3	7	8	5	16	4	15	11	36	32	30	26	20	14	12
3			3	4	2	0	1	19	24	19	2	10	26	35	14	29	42
4				1	7	4	17	19	28	17	34	37	20	7	35	20	0
5				0	5	13	10	21	16	16	10	30	27	30	49	26	19
6					9	8	2	8	13	21	11	31	41	14	12	47	49
7						8	4	1	24	21	35	6	16	28	14	39	16
8						16	0	7	8	0	18	9	7	40	17	29	12
09							6	5	28	10	14	3	7	24	7	0	4
10								7	17	8	28	0	20	41	14	26	23
11								13	6	20	3	32	18	40	43	37	7
12									22	28	14	2	22	9	1	57	55
13									13	16	15	30	38	39	14	28	6
14									24	11	32	14	24	42	36	23	47
15										26	2	36	27	1	47	12	30
16											6	28	5	43	23	25	21
17											0	33	6	8	15	57	50
18											23	36	2	30	19	30	59
19												29	28	35	35	11	60
20												7	41	33	29	1	51
21													9	8	43	40	9
22														43	11	25	48
23														11	18	7	38
24															9	20	25
25															14	8	19
26															23	44	13
27																44	29
28																1	29
29																40	44
30																	1

Thus $A_{i+1} \equiv A_{p-i-2} \bmod p^*$. From (7), we have

$$\lambda_p(i) \equiv \lambda_p(j) \mod p^* \qquad \text{if} \quad i+j=p-1$$

and again by (7),

$$A_{pt+i} \equiv A_{pt+j} \mod p^* \qquad \text{if} \quad i+j=p-1.$$

We append Table I of the $\lambda_p(j)$; by Proposition 4, we only need to give them for $0 \leqslant j \leqslant \frac{1}{2}(p-1)$.

References

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